

Developing and Refining a Learning Progression for Matter: The Inquiry Project: Grades 3-5

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Introduction

A Learning Progression for Matter (LPM) is an attempt at characterizing the evolution, from infancy to 12th grade, of children's and adolescents' knowledge in the domain of matter. In an initial paper on LPM, Smith, Wiser, Anderson, Krajcik, & Coppola (2004, 2006) mapped this domain by working "backwards" from the (simplified) atomic-molecular theory which is part of the American high-school curriculum. They asked "What prior knowledge would allow students to understand the atomic-molecular theory?" They concluded from a review of the literature about students' difficulties with the atomic-molecular theory, and innovative curricula aimed at remedying those difficulties that, while the sources of some difficulties lay in the atomic-molecular theory itself and require innovative approaches to teach it, others are to be found in students' understanding of matter at the macroscopic level and in their epistemology, which are typically not sufficient to support a meaningful understanding of the molecular-atomic model. Thus their LPM proposed building a set of "big ideas" about matter at a macroscopic level during the elementary school years, which included important ideas about measurement and modeling, to serve as an appropriate foundation for learning about the atomic-molecular theory of matter in the middle and high school years. They also proposed a set of "learning performances" that could be used to chart students' progress. In a later paper, Wiser & Smith (2008) extended the analysis back to infancy and the preschool period, discussing many of the same issues and studies through the analytic lens of conceptual change.

The Inquiry Project is an NSF funded LP Project¹ that is specifically concerned with elaborating on a LPM for children in grades 3-5 of elementary school. Building on prior LPM work, it is concerned with further specifying how children's macroscopic concepts of material, matter, weight, volume, and density might evolve during the elementary school years with supportive curricular units, designing such curricular units as well as more in depth clinical interview assessments, and studying the progress elementary school students make with these concepts over a three year span of time.

This paper draws on some aspects of current work of the Inquiry Project to address four key questions that are of broad importance for learning progressions work:

- What **units of analysis** need to be part of LPM work? How are multiple units of analysis related?

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- What **aspects of students' starting knowledge** are salient and motivating and should be focused on in curricula because they have the greatest payoff in promoting change?
- What **representational tools** provide important sources of coherence and inference as students revise and further their ideas about matter?
- By **what criteria** can LPs be **evaluated and revised**?

Several central questions about our approach need to be addressed, or at least acknowledged—Are there multiple viable LPM and if so what would they look like? How radically different could they be? How early is it useful to introduce ideas about atoms and molecules? Can explicit ideas about matter be entirely co-constructed with ideas about atoms and molecules, or are we right in assuming certain basic epistemological and conceptual understandings need to be developed first?

These questions, along with those this paper addresses more specifically, will help us, and others in the “LPM enterprise” modify our theoretical approach to LPM, and how we use it to design and assess curricula. It is well beyond the scope of this paper to answer them but a brief presentation of *how “generic” a LPM is* might be provide a useful context for our answers to the four specific questions above.

Of course, the evolution of any kind of knowledge is entirely (but not uniquely) dependent on learning experiences. In that sense, if LPM were taken to represent the evolution of individual children's knowledge in all its details, there would be as many as there are children. On the other hand, LPM is about children's and students' conceptual framework, not specific beliefs about specific objects and specific materials. E.g., it is not relevant to LPM that some children know something about plastic while others know something about steel; what is important for LPM is whether they think that, once ground plastic is still plastic and steel is still steel, etc. The question, then, is: Can this conceptual framework evolve in more than one way, depending on young children's experiences and the instruction students receive in school?

Starting with young children's knowledge, it is likely that innate constraints on perception and early conceptualizations, combined with a physical world that affords the same knowledge about objects and materials to every child, makes our characterization of the early part of the progression apply to all children (except of course in case of extreme deprivation or abnormal development, and allowing for different rates of development and differences in the richness of individual children's knowledge. The range of knowledge about objects and materials children bring to kindergarten varies greatly but the concepts organizing this knowledge do not.

What of LPM for the K-12 range? We acknowledge that our LPM was developed mostly using data about students in American, European, Australian, and Israeli schools. Nothing in the research from other countries that we are aware of, indicates that LPM does not apply to students in a large number of ethnic and linguistic communities. However, notably absent is research about Asian or Russian students, whose communities have a tradition of innovative and strong mathematics education; or of students in Africa or South America, or of entirely unschooled children. Certainly, systematic cross cultural studies would provide much needed data about this issue.

Returning to students in American schools what does LPM represent vis-à-vis the evolution of knowledge of individual students? We believe that LPM is unique at a relatively large grain (both time-wise and knowledge-content wise). One could say metaphorically that concepts have a life of their own. The knowledge network in one age range strongly constrains the processing of new information and therefore the conceptual changes that can take place at that age. Cognitive processes further constrain the assimilation of new information into the knowledge network. Thus it is likely that concepts don't have much "wobble room" as they evolve from the way preschoolers think about matter to a way of thinking about matter consistent with the scientific view. This does not mean every student will achieve the end state; on the contrary, there is ample evidence that most students don't. But given the right instruction, they could.

More specifically, we advance three empirical claims: 1) students cannot achieve a scientific understanding of the Atomic-molecular theory, without a scientific understanding of matter at the macroscopic level; 2) achieving a scientific understanding of matter at the macroscopic level requires a major reconceptualization of a previous state of knowledge about matter, typically that held by the majority of late elementary school/middle school students; and 3) insuring this reconceptualization takes place for the majority of students requires curricular innovations. The ongoing work of the Inquiry Project will be directly examining the later two claims.

Units of analysis

The Inquiry Project is concerned with a major **reconceptualization** that occurs as children come to construct and inter-relate the concepts of **weight, volume, material, matter, states of matter, phase change, and density** in a physical theory of matter that is consistent with the expert theory. The youngest children participating in the Inquiry Project are third graders; most of them enter third grade with a knowledge network about matter which is incompatible with the expert theory. We will refer to this knowledge as the lower anchor². The concepts in the lower anchor are either different in fundamental ways from their expert counterparts (weight, melting, material), not clearly differentiated from each other (weight and density), or absent (volume, gas). We are interested in understanding how we can teach children to build on their (lower anchor) concepts and knowledge to develop understanding of matter at the macroscopic level that is compatible with the expert theory (upper anchor). Upper anchor knowledge and concepts are valuable in and of themselves, but also are necessary to understand the atomic-molecular theory.

Fundamentally this reconceptualization involves **conceptual change**. To understand conceptual change, one needs to consider (and inter-relate) multiple units of analysis: **concepts,**

² We borrow the concept of "lower anchor" and "upper anchor" from Anderson and Mohan. The lower anchor is the state of knowledge a majority of students have as they enter the grade range of interest. In the Inquiry Project, the lower anchor is the knowledge most third graders have at the beginning of the school year. The upper anchor is the targeted knowledge at the end of the grade range of interest. In the Inquiry Project, the upper anchor is the knowledge we hope students will bring to middle school.

“subconcepts”, beliefs and relations between them. One also needs to consider the real world **situations** to which the concept applies and the ways the child **symbolizes** the concept (Vergnaud, 1996).

At one level, concepts can be thought of as mental symbols, roughly the grain size of a word, that are part of larger conceptual fields (networks of concepts and beliefs about the relations of those concepts) (Carey, 2000). The purpose of these conceptual fields is to organize and make sense of some domain of phenomena in ways that capture important generalizations about the domain and that allow communication about, as well as prediction and explanation of, observed events in the domain. Two conceptual systems may differ in how they “cover” a domain: what units are distinguished (and lexicalized as single words) and the ways they are inter-related in talk and patterns of explanation.

For example, we will argue in the lower anchor children use one undifferentiated concept (weight/density) where a later system distinguishes (at least) two (weight and density). More specifically, children will say both “This box is heavy,” “steel is heavy,” “things sink because they are heavy,” and “heavy things make the scale go down.” By using heavy is all these contexts, they indicate that a common (entity) (variable) applies to all those situations. In contrast, later they can distinguish situations where density is relevant (things sink if they are denser than the medium in which they are immersed; steel is dense) and situations where weight is relevant (this box is heavy; the heavier object makes the scale go down). Hence the classes of situations recognized as similar shifts.

Viewed in this way, concepts are constituents of more complex representations, such as beliefs. Beliefs can be of a variety of sorts: specific beliefs (This box is heavy), broader empirical generalizations (Big things tend to be heavy) or explanatory beliefs (This box falls because it is heavy.) Particularly important are beliefs that express **ontological**, **epistemological**, and **explanatory** commitments or that are organized **in mental models**.

Ontological commitments are one’s beliefs about what sort of entities exist in a domain and what their most essential aspects are. In the lower anchor, ““having weight” means “feels heavy---iaffects ME by pushing/pulling on my hand.” This has an **epistemological** aspect as well—the way I know something has weight is by picking it up to see if it feels heavy, which is part of a broader epistemology that privileges first hand observation, over testimony of others, or use of measuring instruments. (Although, I can be told by someone else that something is heavy, I am much more convinced when I pick it up myself.) Ontological commitments also reflect one’s deepest **explanatory principles**. In the lower anchor, I know something is heavy **because** it feels heavy. There is no level of deeper explanation of weight than this.

Of course with increasing knowledge, one’s **ontological, epistemological and explanatory commitments may change**. For example, in a later system “having weight” may mean having some amount of matter that pushes on things (whether or not it is detectable by me). This can involve important shifts in epistemological commitments: the way I know about weight is by using scales or other instruments to measure it, that can vary in their sensitivity and reliability. In turn there can be shifts in patterns of explanation. Weight, which initially was used to explain other events but itself was unexplained, is now seen to be a function of the

amount of matter in an object, which in turn can be predicted from knowledge of the density of the material it is made of and its volume.

Beliefs themselves can be inter-related in a variety of ways. Some beliefs are mutually supportive as was illustrated with the case of one's ontological, epistemological, and explanatory commitments. Such tightly related beliefs are very resistant to change. Some may be explicitly related **in mental models** that apply to a class of situations. Some may even become tightly organized in a **formal theory** for a domain. We do not think young children structure their beliefs in formal theories, but we think they do have some ontological, epistemological, and explanatory commitments. They may also use mental models.

Different beliefs are **evoked by different contexts**³. For example, if a student is engaged in weighing objects of different sizes that all made of steel, the belief "bigger things are heavier" is evoked while the belief "Steel things are heavy" is unlikely to be. Some of the cuing of different beliefs may be based on salience of certain features in a situation. Part of the art of using concepts is knowing what beliefs are most relevant to consider in different situations. However, this context dependency in activation of beliefs also means that potential conflicts among different beliefs involving the same concept may fail to be noticed by the students (and scientists) alike.

At another level, concepts are themselves complex mental representations, having both an implicit (e.g., sensori-motor and context aspect) as well as more explicit and symbolic aspect. Keil & Lockhart (1999) have argued that they are heterogeneous structures having both an associative and explanatory component, and that these components are constantly interacting.

Concepts may be symbolized in different ways that affect how they are combined and the inferences that are made as one uses them. For example, in the lower anchor, weight can be symbolized with words (e.g., "heavy" and "light") as well as using analog magnitude representations (heavier things have great analog magnitudes). Analog magnitude representations capture some aspects of weight—its a core comparative structure (if X is heavier than Y it is represented with a greater magnitude) but not others-- such as information about how much heavier X is than Y. Later children can extend their ways of symbolizing weights, including mapping to discrete numbers (X weighs 5.2 grams). They also use weight line representations, which explicitly capture the extensivity of weight as well as its ratio structure.

Thus, one's knowledge network is complex with multiple inter-acting elements. The meaning of a word such as "heavy" is not localized in any one belief, but in large collections of beliefs that can shift over time. When we compare knowledge networks on a large time scale—for example pre-schoolers' knowledge and late elementary students' knowledge, or the macroscopic knowledge of the uppermost anchor to the knowledge of most late elementary students, we find clear differences in terms of our units of analysis--in ontological commitments, in explanatory beliefs, in relations among beliefs and which beliefs are core, in the salience of beliefs over others (e.g., the belief that heavier objects feel heavier and the belief that heavier

³ We are not specifying whether "evoked" implies that beliefs reside in long term memory or that beliefs are created, from other mental elements, when students engage in prediction, explanation, etc. We think we can present our units of analysis without engaging in this debate.

objects make the scale go down are present in early and advanced knowledge networks but, if they are put into conflict, hefting “wins” over reading the scale early in LPM, whereas scale reading “wins” over hefting later in LPM.), etc.

This does not imply that, when reconceptualizations happen, they happen instantaneously, on the contrary. The Inquiry Project curriculum is designed to foster stepwise changes, designed to amount to a radical reconceptualization once we compare children’s state of knowledge at the end of fifth grade to their knowledge at the beginning of third grade. Planning the curriculum requires orchestrating classroom activities which will change parts of the knowledge network without destabilizing it, and build up links between those partial changes, first within concepts then between concepts.

To sum up, characterizing knowledge in terms of concepts, subconcepts, and beliefs, and the contexts that evoke those beliefs, allows to capture meaningfully the evolution of students’ knowledge at several time-scale—“day-to-day” (how did classroom activity A affect their interpretations of those contexts X, Y, and Z?; from one curriculum unit to the next (how did Curriculum Unit 1 affect their concept of weight?; from one grade to the next (how did their knowledge about matter change)? and at larger time scales as well.

This way of thinking about learning progressions as involving a series of reconceptualizations has implications for the design of curricular units and assessment. For example, it means the assessment of concepts is complex and cannot be accomplished with any single task. Nonetheless, one can design tasks that target particularly important beliefs about concepts (e.g., ontological, epistemological, and explanatory beliefs) and analyze responses in ways that look at larger patterns of beliefs. Unfortunately, traditional school assessments rarely ask questions that tell us about children’s underlying ontological, epistemological, and explanatory commitments.

The next sections illustrate how our units of analysis guides curriculum design. Whether or the Inquiry Project curriculum is successful at fostering the targeted reconceptualization is an important test for the validity of our units of analysis.

Lever concepts

When designing curricula it is important to identify concepts that are a **salient** part of students’ everyday thinking and **densely connected** to other ideas, so they offer many **points of contact with instructional material** and therefore multiple **sources of conceptual change**. The Inquiry Project has dubbed such concepts “*lever concepts*.” Once a lever concept has been partially restructured, it can be involved in restructuring other concepts, or in constructing concepts that do not exist in the lower anchor, propelling students’ knowledge network forward. Moreover, partial reconceptualizations can facilitate other partial reconceptualizations within concepts. Working on change in lever concepts first thus provides the “most bang for the buck”.

In the Inquiry Project, we have identified **weight, size, and material** as lever concepts that are central to the (later) development of other related concepts: **volume, density, and matter**.

Here we discuss weight and material to illustrate lever concepts' two key characteristics:

- (a) They are already present in the lower anchor but need to undergo reconceptualization themselves;
- (b) They are salient and densely connected to other concepts; hence they provide multiple entry points for new learning and, as reconceptualization occurs, change can be propagated (or spread) throughout the network

A. Lever concepts are already present in the lower anchor, but need to undergo important reconceptualization (if the curriculum is to be effective).

The lower anchor concept is a **precursor** of the higher anchor concept in the sense that some of the components of the later concept are present in the earlier one. For example, very young children know the word “heavy,” which they associate with a downward pull or push on their body (e.g., while hefting objects or trying to lift them) and for most students in the early elementary grades, deciding whether object A is heavier than object B is best based on hefting the objects. The “heft” sense of weight remains part of the adult concept, although adults know that hefting is both imprecise and at times misleading.

It would be wrong to think that the heft sense of weight is simply combined with new knowledge (e.g., about measuring weight with a balance scale). “Combining” suggests enriching existing knowledge. In the case of lever concepts, **integrating new knowledge into the lower anchor network implies and amounts to reorganizing the parts of the network relevant to the concept**, rather than simply enriching the concept with more elements.

For example, associating “heavier” with “side of the balance scale that does down” and with “feels heavier when I heft” would cause conflicts if simply added to the lower anchor network—a small piece of steel may feel heavier than a big piece of wood, but actually be lighter. Before reconceptualization, some students simply ignore the conflict—the object that feels heavier *is* heavier and “I don’t know how a scale works.” Others may try to explain why the “lighter” object makes the scale go down (e.g., by invoking size as affecting the scale as well). After reconceptualization, however, students express trust in the scale over hefting, may attempt to account for why the larger object is actually heavier (“although it is made of wood, it is a lot bigger”), and may attempt to explain why the unaided senses can be deceiving. These students haven’t simply “learned to use a scale.” They have constructed an objective concept of weight, know it is a property of amount of); relate it to other physical entities (it is a function of material and size), and are using a new epistemology--privileging measurement over sensory information. These changes are interdependent and amount to a reconceptualization which includes an ontological shift—weight is now a property of the material an object is made of, not of the object per se (See Linchpin, below).

Even before children can explain away the conflict between felt weight and scale readings, and before they differentiate clearly and explicitly “heavy” from “heavy for size,” this new cluster of interrelated beliefs about weight (i.e., scale measures weight, weight is related to the amount of material) can be called upon to reason with as a unit; it forms a “sub-concept”—“scale

weight.” This sub-concept is consistent with the scientific concept, and therefore enters in many powerful generalizations and has more explanatory value than the lower anchor concept. This coherence, which students will discover progressively, will serve to further foreground it among other parts of the students’ weight concept.

Material. Material also is present in the lower anchor as a precursor concept, but again not yet in a form that is useful as a scientific concept. Materials are an important aspect of the physical world for young children, as they affect their interactions with objects. Children are familiar with the appearances and behaviors of some materials—glass breaks, plastic does not; rubber things bounce; steel is “heavy;” wood burns, butter melts; ice is frozen water, trees are wood. But they do not know the word “material;” do not know what general properties distinguish one material from another, and have no generalizations about material as a category. Although most eight-year-olds know that, when a wooden spoon is cut into chunks, it is not a spoon any more but the pieces are still wood, many believe that sawdust is not wood, and are not sure whether it would burn. Moreover, they are less sure material X is still material X when it is ground, especially if it is unfamiliar or changes color, and even less sure of it when a solid melts. Thus material is not yet an ontological category.

Similar to weight, a new concept of material needs to be slowly constructed; this involves reorganizing rather than simply enriching the existing one. As for weight, part of this reorganization is creating a new ontology—the ontological categorical material (as a specific form of matter distinguishes from other forms of matter by specific properties—density, hardness, and melting point among others), which depends on an epistemological change—perceptual attributes are not necessarily good indicators of physical properties.. In the same way that the new concept of weight is a property of amount of matter, independent of the object’s shape, and objectively measured rather than perceptually assessed, the new concept of material applies to any part of a (homogeneous) object instead of being a collection of perceptual attributes of whole objects and therefore inherently linked to matter. This in turn allows differentiating between material identity and physical state and therefore between different materials and the same material in different states (if the concept of material is inherently linked to matter and distanced from perceptual properties, it is easier to believe that melted X is still X even if it does not look like X in solid form). As with all reconceptualizations, the reconceptualization of material has a semantic aspect, as well as a conceptual aspect. For example, some powders have their own names (“sawdust”) while other don’t (“iron filings” “gold powder”).

One can see that those different aspects of the reconceptualization of material are not independent of each other; for example, understanding the distinction between the identity of a material and its physical state is closely linked to understanding “made of” as “composed of.” In addition, aspects of the reconceptualization of material are related to aspects of reconceptualizing weight as students relate weight to amount of material, differentiate weight and density, and know that density is a distinguishing property of materials.

B. Lever concepts are salient in the lower anchor and densely connected to other ideas;

they offer many points of contact with instructional material (“entry points”). As reorganization among lever concepts occurs, change can spread throughout the network.

In the section above, we argue that changing the lower anchor concepts to the upper anchor concepts involves modifying many aspects of the lower anchor concepts and beliefs that are interrelated. A challenge for teaching is how to help student “break into” this new system of ideas. Clearly, new relations cannot be introduced all at once; so careful choices need to be made about sequences and ordering. These choices have important consequences—some may destabilize the system while others propel it forward. Thus we argue that decisions about that sequence and ordering of classroom activities can’t come from analysis of the expert system alone; it is critically important to consider the form and organization of ideas in the lower anchor.

Because lever concepts and words for symbolizing them already exist in the lower anchor, they provide meaningful entry points for all students. For example, one “entry point” for weight is that hefting “measures” weight. A second entry point is that balance scales respond to the weight of objects (because objects press down on the scale pan as they do on one’s body). Presenting objects made of different materials (such as the density cubes, discussed below) is another entry point for discussing the properties of objects and materials. In addition, because these concepts are richly connected with many other concepts, when lever concepts “move” toward the upper anchor, they contribute to moving other concepts along, or constructing new ones. In contrast, other concepts in the expert network (such as volume, density, matter) are not yet distinct concepts for students in the lower anchor nor symbolized with individual words (although components of these concepts may be part of other precursor concepts). Leading with these ideas would not only involve leading with new vocabulary, but would also potentially destabilize the system rather than propel the system forward. Unfortunately, much of science education involves prematurely introducing new ideas and symbols without consideration of how they will be understood by students. A learning progressions approach is committed to thinking through sequences that work to move the network forward while preserving intelligibility to students.

Focusing initial explorations on concepts that are already present in the lower anchor does not mean that one works on concepts in isolation, completes work on one concept before moving to the next or refrains from introducing students to new ideas or forms of symbolization. Instead, one always is considering portions of several concepts (foregrounding some, backgrounding others—see Text box below), working on successive subconcepts, such as scale weight, each of which involves relations among parts of concepts, revisiting concepts and amplifying the subconcepts and contexts considered. Further, new forms of symbolization are critical to the reconceptualization of the lever concepts themselves. Significantly, these forms of symbolization (which we describe more extensively in the next section as LINCHPINS) are not typically exploited or used in the traditional curriculum.

Foregrounding and background in the Inquiry Curriculum

Overall, in the Inquiry Project, the five core concepts are present in each unit, although some are foregrounded and taught explicitly while others are implicit. For example, weight and material are presented front and center in grade 3, while volume is introduced only briefly at the end of the unit; density and matter remain implicit. One of recurring props for hands-on activities are a set of “density cubes:” same size cubes made of different materials with different densities. In grade 3, they are used to compare the merits of hefting vs. using a scale to measure their weights, and to systematize the set of properties that can be used to compare materials (one of them being the weight of the cubes). Neither the concept nor the word *density* is used but students’ attention is drawn to the cubes all being the same size, and the teacher occasionally uses the term “heavy for its size.” Thus density starts being scaffolded in third grade, linguistically, as a placeholder, and by being embodied in the set of equal size cubes. We see this kind of “implicit scaffold” as part of helping students to be ready for a new idea later on.

In the 4th grade, the focus shifts to a variety of Earth Materials, which includes liquids, granular materials, and solids. Volume and material are front and center, while weight recedes, as students investigate water displacement and discover it is volume, not weight that is relevant. Students explore multiple senses of volume and also begin to distinguish heavy and heavy for size as they compare samples of different materials that are all the same weight and observe their very different volumes.

The five concepts will once again be present in the fifth grade, with different saliences--greater foregrounding of matter (as students consider gases and the process of phase change) and weight (as they consider conservation of weight across phase change).

In the next section, we turn to a more detailed consideration of how levers and linchpins work together in the Inquiry Project curriculum to propel the network forward.

Linchpins: Overview

As previously discussed, Lever concepts exist in the lower anchor, although only as precursors of their counterparts in the upper anchor (e.g., weight, material, volume). They are the concepts that are most amenable to initial restructuring and ontological change, and most likely to contribute to the reconceptualization of other concepts via content relations. For example, differentiating “heavy” from “heavy for size” allows “objects made of material X are heavier for their size than objects made of material Y,” the first step toward constructing the concept density.

In contrast, Linchpins are not themselves concepts; they are “organizers.” They express structural aspects of concepts and/or relations between concepts. At the same time, they are tools that make reconceptualizations possible; without them, hands-on experiments and observations would enrich the knowledge network in the lower anchor, but not propel it forward toward the upper anchor.

Questions about generalizability. . Everyone in the Symposium has rightly voiced that linchpin is not a clear construct. We hope that the symposium, and the co-authored paper we are planning to write, will contribute to clarifying the issue of linchpins. In what follows, we present and characterize the linchpins for the Inquiry Project; it is **not clear yet to us how this characterization generalizes** beyond the elementary grades range; beyond our LPM; to other LPs, etc. Linchpins, as defined below, seem to apply to other physics domain (e.g., heat and temperature). We suspect it might apply in other domains, and therefore to other LPs, once transposed to apply to other kinds of theoretical terms, or to formal aspects of theoretical terms other than quantification, but have not explored that possibility yet. Linchpins at latter points in a matter LP may also take quite different forms from the ones described below; but if the construct is to be of use, we believe there need to be some constraints on the form they may take.

Linchpins

The linchpins in the Inquiry Project are: the **weight (and volume) measure lines**, the **compositional model of matter**, and **dots models**.

Linchpins are **visual representations** of the **quantificational structures** of the **upper anchor concepts** (e.g., extensivity of weight and volume; relation between sweetness, amount of sugar, and amount of water). Thus, by definition, they do not exist in the lower anchor. They **represent** quantitative **properties** of physical entities.

Linchpins are **models** that capture some important **structural aspects** of physical quantities. As in all models, physical entities are represented symbolically in linchpins; but, unlike most scientific models, linchpins do not represent the causal interactions among physical variables (as, for example, White et al's models of force and motion and electricity do); instead, they express some important characteristics of the quantification of physical variables.

Linchpins are constituted, during classroom activities, from knowledge elements in different domains--not only the domain under study, matter, (or more broadly the domain of physical objects and phenomena) but also the numerical and mathematical domain, and possibly, via analogy use, other domains. Moreover, once mastered in the context of one variable, they are easily used to represent other physical variables which share the same quantification structure.

Thus linchpins **“hold ideas together” within and across certain concepts**. Within a concept (e.g., weight) they make available several aspects of its quantification, and their relations (e.g., units of weight have to be equal; the weight of an object is the number of weight units that equals it; the weight of two objects together is the sum of the weights of the objects; if one keeps dividing a chunk of matter, its weight decreases but never goes to zero.)

By applying to several variables, linchpins make available the quantificational structure of those variables at a more abstract level. This is similar to the role of analogies in reasoning: when the same analogy applies to different situations, the common structure of those situations

becomes apparent. This is not a coincidence—models share many important features with analogies.

Linchpins are **important tools for reconceptualization**. As all scientific models linchpins are sources of inferences and discoveries about the physical variable they represent. And as with all models, the linchpin and the concept are co-constructed through successive mappings between visual representation (linchpin) and concept; the concept evolves as students make sense of the linchpin and its relation to the concept, and discover new affordances of the linchpin.

Once mastered in the context of one concept (e.g., weight), a linchpin can be more easily used in the context of another (e.g., volume).

These ideas bear definite resemblance to the role of Piaget's logical operations in conceptual development but also differ from them in several crucial ways; one of them is that **linchpins are domain specific** (or more exactly, are constructed and used first within a specific domain)

Weight measure line

Weight measure line from the expert point of view. The weight measure line expresses the quantification of weight, i.e., how weight varies with amount of matter. Weight is an extensive property of matter—doubling the amount of matter doubles its weight; concatenating three identical objects results in an object with triple the weight. Weight is also gravitational force, so that the equality of the weights of two objects can be established with a balance scale. That is why the weight of an object (let us call it A) can be measured by establishing its equivalence to the weight of a set of identical objects (the weight of each of these objects is a weight unit) via a balance scale and counting the number of weight unit objects. The operation of physically concatenating a set of weight unit objects can be represented by numerical addition of 1's. Therefore, the weight of A can be represented by the cardinality of the sets of weight unit objects—i.e., by counting the weight unit objects. The weight of the object resulting from concatenating two amounts of matter, A and B, can be computed by adding the numbers representing the weight of A and the weight of B. And similarly the effect of removing an amount of matter from another one can be computed via subtraction. Thus, the number line can be used as a model or visual representation of weight and of the weight transformations resulting from concatenating, dividing, or removing amounts of matter.

Elements that can be recruited from students' lower anchor to construct, use, and learn from a weight measure line. Most eight year olds do not know yet that weight is an extensive property of matter; most of them do not know it is an intrinsic property of matter at all⁴. They typically believe tiny things weigh nothing at all; and that some materials (e.g., Styrofoam) have no weight. These aspects of the lower anchor knowledge are related to each other, and centrally related to the ontological belief (a belief that grounds what they think weight is)—the weight of

⁴ Weight is not really an intrinsic property of matter but one can say that it is wherever there is a gravitational field.

an object is how hard it pushes or pulls on one's body. Achieving an understanding of the extensivity of weight and the nature of good measurement are two main goals of working with weight measure lines.

Given that the core meaning of “heavy” in the lower anchor is pushing down on other things, using a pan balance scale to establish whether two things have the same weight, and if not, which is heavier, is intuitively obvious to young students. They also know that two identical things are heavier than one and, more generally, that adding stuff to an object increases its weight (as long as the piece you add is big enough). In other words, they have a qualitative sense of the extensivity of weight, which is (importantly) limited by their belief that very small things don't weigh anything at all, and therefore that adding small amounts to something doesn't change its weight.

Eight year olds also know about integers; have an implicit understanding of cardinality—the last count word represents the cardinality of a set, i.e., tells you “how many objects there are”;— know that, if you combine two sets of objects, you can compute how many there are by adding the numbers of objects in each set; and can represent integers on a number line.

All these pieces of knowledge in the lower anchor can be organized and **coordinated** by involving students in **measuring** weight using a pan balance, and **representing** their measurements on a weight line. The weight line can then be used to support **computations** and **inferences**. **Coordination and inferences are sources of genuinely new knowledge** about weight and of its **reconceptualization**.

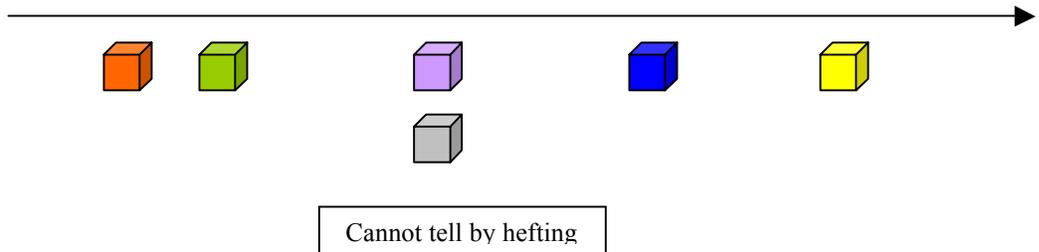
Constructing a weight line. Linking Felt Weight and Scale Weight. Initially, the weight line is an actual linear array of objects (density cubes) according to increasing felt weight. (Step 1 in Figure below.) Uncertainties about some pairs of cubes motivate the use of a balance scale. Students readily relate the functioning of the scale to their own hefting, a similarity, which helps establish the pan balance as a good means of comparing weight. As they realize that the balance scale is both more sensitive and more reliable than hefting their focus switches easily to arraying objects according to comparisons with the balance scale; the concept of weight, at this point, is enriched with “the scale hefts more accurately than I do.” The weight line displays weight ordinally, without numbers. (Step 2. in Figure below).

The next investigation ups the ante--children move from the question of which cube is heavier to the question *how much heavier*. This question cannot be answered without quantifying weight. More problematically, the question itself cannot be meaningful without conceiving of weight as an extensive quantity. The lower anchor concept of weight lends itself to qualitative comparisons only—A is heavier than B which is heavier than C. For “the weight of B is twice the weight of C” to be meaningful, weight has to be conceptualized as a ratio variable (i.e., weight units and weight = 0 must have meaning). Moreover for “the weight of B equals the weight of A and C combined” weight needs to be conceptualized as an extensive variable.

How can classroom activities revolving around a question students cannot really understand, be productive? They would not be without the weight measure line.

Measuring weights with weight units and using the weight measure line. In a series of activities using the balance scale and non standard units—plastic bears, paper clips, and washers—students discover the need for a *uniform* and *shared* system of weight units. (Step 3 in Figure below). Standard units are then introduced, which students use to measure the weights of the density cubes. Students place the density cubes along the weight line, according to their weights in grams. (Step 4 in Figure below.). They can now discuss *How much heavier (or lighter) is one object than another?*

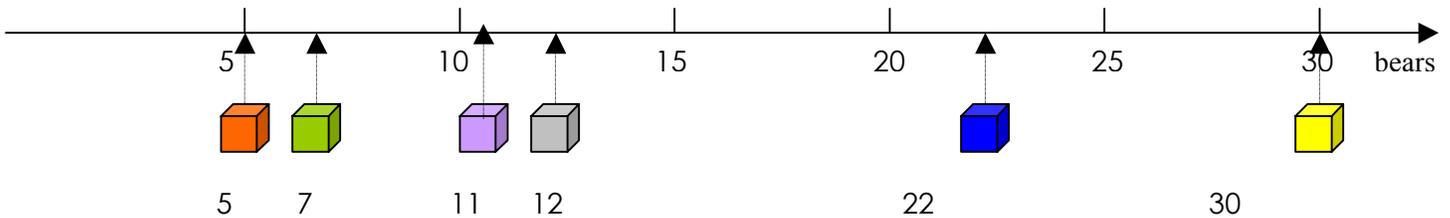
Step 1. Ordering by hefting



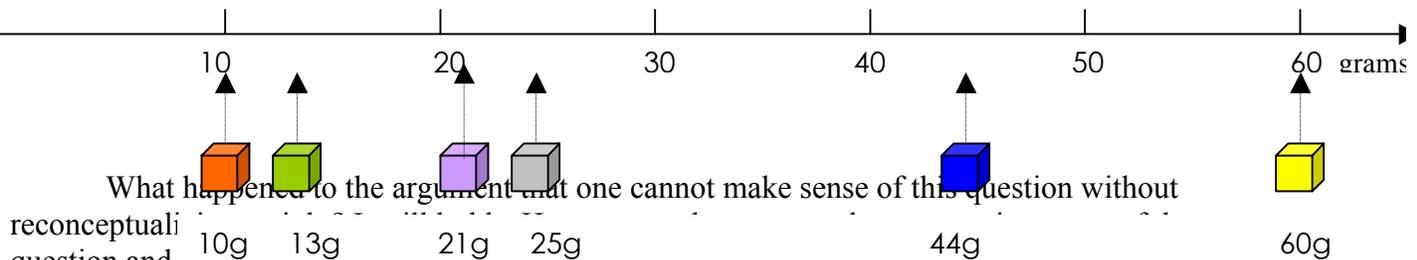
Step 2. Ordering by comparing objects with the balance scale



Step 3. Measuring weights with a balance scale and plastic bears



Step 4. Measuring weights with balance scale and standard units



What happened to the argument that one cannot make sense of this question without reconceptualizing the question and

they move back and forth between measuring weight with the balance scale and using the weight line. Going back and forth between the world of objects and the visual representation results in an increasing understanding of weight as quantifiable and extensive, and of the weight line as a weight measure line embodying those properties.

Measuring the weight of a density cube with a balance scale and non standard units can be understood with the lower anchor concept. How many plastic bears have the same weight as the cube, i.e., how many bears does it take to balance the cube? are meaningful questions for a student who holds the lower anchor weight concept. The “weight” part of the question (making the weights equal) is qualitative. The answer—“7 bears”-- is quantitative but it is not weight that is quantified (yet), it is the number of bears with a weight equal to the weight of the cube.

Number of bears can be displayed on a *number line*, a process familiar to young students. But the line is already a *weight line* because it has been used to represent the qualitative ordering of the weight of the cubes. Without knowing it explicitly, students are *blending* (in Fauconnier’s sense) a number line and the qualitative weight line, i.e., they are applying the properties of numbers (at first, of integers), to weight.

A first reconceptualization involves the weight line and language. The teacher shifts the emphasis from “How many bears have the same weight as the cube?” to “What is the *weight of the cube in bears?*” and then “What is the weight of the cube? “Seven bears.” This shifts “bear” from being an object with a certain weight to being a weight unit. As the meaning of “bear” shifts, students are developing a sub-concept of weight around the belief that weight can be *measured with a balance scale*; it can be assigned *numbers*. This sub-concept is weight as an objective, quantifiable property. It is compatible with the rest of the lower anchor concept--the blue cube felt heavier than the green cube; it made the scale go down when they were placed in the balance pans, and now, its weight in bears is greater.

Once this first reconceptualization has taken place, using a balance scale to measure weight in *grams* is a meaningful activity. (Without being scaffolded by using non standard units first and without the weight line, the same activity may have remained a routine with shallow meaning.)

Drawing inferences from the weight measure line. Students can start exploring new questions. Are there objects that are just one gram apart in weight? Can objects be less than a gram apart? How many? The fact that there are other weights “in between” any two weights on the measure line becomes more apparent as children imagine cubes made of other materials (clay, soap, stone, glass, brick, concrete), and speculate where their imaginary cubes might go on the line and why. These activities enrich the new sub-concept—objective quantifiable weight--and the meaning of weight units. They also help develop the notion that weight is a continuous variable (an issue we will not develop in this paper).

The next question--How much weight would need to be added to the oak cube to have it be the same weight as the pine cube?—can be answered empirically, using the balance scale. Place the pine cube in one pan, the oak cube in the other pan and see how many grams need to be added to the pine cube to balance the scale. It can also be answered by using the weight line.

How to do so is not obvious and is challenging for many students. The isomorphism between physical actions with cubes, grams and balance scale, and counting line segments and reading marks on the weight line gives the weight line its “measuring” meaning. In other words, students develop the implicit understanding that the conclusions one reaches by reasoning with the weight line are true of the real world. They learn to validate the weight line as a model of weight.

More generally, students will internalize the structure of the weight line as the quantificational structure of weight. One weight unit is represented by one line segment on the weight line. The weight of an object is represented by as many contiguous line segments as there are grams balancing the object on the scale. When one reads “the weight of this object is 15g” from the mark on the line, one is also taking in that it is the sum of 15 weight units.

More inferences from the weight measure line; more discovery about the relation of weight and material. As students become accustomed to thinking of weight along the weight line, interesting questions arise about smaller and smaller pieces of material. For instance, many students do not immediately understand that there can be values on the weight line between say, 3 and 4 grams, or more interestingly, between 0 and 1 gram. The visual representation of weight makes this idea graspable, especially if it is linked to the idea that a more sensitive scale would discriminate between these different weights.

Can one make pieces of stuff small enough that they stop weighing anything? Students break a 4g piece of Playdough in two, placing one of the pieces on the 2g mark, and keep going, toward the origin of the weight line. This leads to discussion, “Will you ever reach 0?” Do you think a tiny piece could weigh nothing at all?” The weight line is part of the argument. Representing 1g with a larger line segment allows them to keep dividing it further... The weight line also remains yoked to the real world—a very sensitive scale would detect a very small piece.

Many students conclude from these (and other) activities that indeed, any tiny piece must have weight.

Weight line discussions also help give meaning to the zero point. For example, we have found it is not initially obvious to students that weight lines start at 0. Some students think light objects weigh less than 0; further when students what consider what happens with repeated division, some confuse repeated division with repeated subtraction and get to negative numbers. Part of the confusion is with students understanding of fractions and numbers. These weight line discussions are an important context for grounding meaning of fractions and operations of division. Further, if weight is tied with amount of matter, it motivates thinking that the smallest amount would be no matter which be nothing or zero.

Relating weight to other concepts. Some elements of a weight line are applicable to other concepts, among them volume. For example, measures of volume can also be displayed along a volume measure line that has zero point, fixed units that can be subdivided into fractional parts. Although there are many other issues specific to volume that need to be worked through in developing its measures, students are aided in the process by having some more general

understandings of measure from their work with earlier quantities. Further, Cartesian graphs combine two measure lines in ways that visually display inter-relations among two measures

Weight Line. Conclusions. The activities and discussions engendered by the weight measure line (potentially) create a new core sub-concept—objective and extensive weight, measured by a scale, and more closely linked to amount of stuff than the concept of weight in the lower anchor. This core sub-concept, the availability of a scale (or of more sensitive ones), the link between dividing an amount of stuff and dividing a line segment on the weight measure line, embodied by placing each piece where it belongs on the weight line, support the idea that even tiny pieces would weigh something, as well as starting to discuss the difference between “feels like it weighs nothing” vs. “weighs nothing”

By foregrounding different features of the weight measure line gradually, aspects of students’ initial weight concept can be both capitalized on and recast to give meaning to the different features of this new representational tool. In other words, a piece-by-piece approach allows students’ initial weight concept to enable the restructuring, not hinder it. Back-and-forth inference between the weight line and weight itself is the vehicle for the progressive transformations.

The weight line’s role in the restructuring of weight qualifies as linchpin for several reasons (a) it links felt weight and scale weight while privileging scale weight and (b) it links the elements of weight measurement to knowledge about number and counting, and to components of the weight concept, via a common visual structure.

Furthermore, it links weight and other concepts (by supporting the construction of the compositional model linking weight, volume and material, and an understanding of general characteristics of good measure that can be applied to learning to measure other quantities such as volume).

Compositional model of material

To apply a compositional model to a chunk of material involves mentally decomposing it into pieces; each piece maintains its identity while, as a group, they keep constituting the original whole. Like the weight line, this model also involves explicit symbolization, although it works with mental symbols and images rather than physical inscriptions.

We hypothesize that a compositional model contributes to enriching the belief in conservation of amount of material during shape change, cutting and grinding. Most elementary school children say that changing the shape of an object (Piaget’s ball and pancake transformation) does not change “how much clay there is” because “you did not add or take away any.” The compositional model supports, deepens, and extends this judgment by providing an explanation—the ball can be thought of as made of pieces of clay, the pieces are simply rearranged into a pancake; their number does not change therefore amount of clay does not change either.

Cutting a chunk of material into pieces, including grinding it, is “the compositional model in action”—the pieces still constitute the whole. Thinking in terms of the compositional model should help students realize that the powder has the same amount of material as the original chunk, although it appears very different.

The compositional is particularly useful as a thought experiment when students are investigating cutting a piece of material again, and again, and again; will pieces get so small that they will eventually disappear? Many elementary school children think so. But when asked to keep the whole chunk in mind—if the pieces all disappear, it is like making the chunk disappear by cutting it—they might change their mind.

The compositional model should also support the conservation of *material identity*. Some early elementary school students know that powders are the “same stuff” as the chunk they came from, but others say it is not the same material because it looks different (e.g., the powder is a different color. Mentally grinding the material does not change its appearance; it may offer support for believing that the powder is the same material as the chunk.

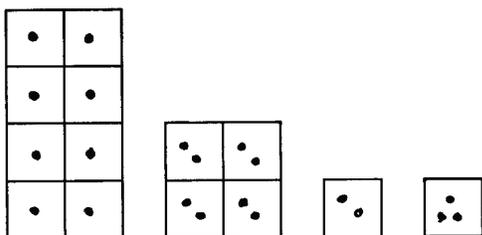
We hypothesize that the compositional model can contribute to the restructuring of the concept of weight as well—the sum of the weights of the pieces has to be the weight of the chunk since all they do is superimpose a “grid” on the chunk. This supports the extensivity of weight, and its quantification. It also supports adopting the belief that all matter has weight—however small the pieces children envision, they still make up the whole; therefore even the tiniest piece has weight.

The same inferences can be drawn about the volume of a chunk of material. Through the lens of a compositional model, volume and weight become quantifiable, extensive, and inherent properties of pieces of any material of any size.

A compositional model can also serve as a stepping stone to schemas of the packedness of particles that can be used for thinking about a broader range of phenomena (e.g., varying concentrations in mixtures, mass being conserved while volume changes when an object is heated).

Compositional model: Summary. The compositional is a linchpin for multiple reasons. First, it can be constructed from three different pieces of knowledge in the lower anchor although it is not contained in the lower anchor—(a) mentally dividing a chunk is like physically cutting it; (b) object permanence (pieces of stuff continue to exist when they are spatially displaced); and (c) number conservation (the number of pieces stays the same when they are spatially rearranged). Second, it embodies the quantification of amount of matter, weight, and volume. Third, it supports inferences which lead to new knowledge (tiny pieces of matter continue to exist, have weight and have volume). Finally, it “holds things together” in several ways—conservation and identity are tied to quantification; the model applies to amount of matter, volume, and weight; and it embodies the principle that volume and weight are intrinsic properties of matter.

Dot models.



Another potential linchpin that may be used in the Inquiry Curriculum (especially in investigations of concentrations) is a Dot Model (see Figure below). Dot models express the relation between two extensive quantities and an intensive one—for example, amount of water, amount of sugar and sweetness; amount of matter, amount of heat and temperature; amount of matter, volume, and density, etc. Dot models are particularly useful not only because they provide an explicit representation of an (unseen) intensive quantity (e.g., sweetness represented as dots per box) that is distinct from the numeric values of the other two measured extensive quantities (e.g., total amount of sugar and water represented as total dots, total boxes, respectively), but also because it shows how the numeric values of these two quantities determines the numeric values of the third. It also shows how something can have the same intensive value (e.g., a mixture can have the same sweetness) in some conditions when there different values of the two extensive quantities (e.g., different amounts of sugar and water).

Conclusions about linchpins

Our three linchpins share several features. They don't exist in the lower anchor. They require well orchestrated instructional activities, focusing on the coordination of the linchpin with hands-on activities. They embody the basic quantification of physical entities. They are co-constructed with concepts: when students explore weight lines as inscriptions, draw inferences from a compositional model, or try to interpret a dot model, they are, at the same time, giving meaning to them as tools of thought, and restructuring their concepts. **It is the mapping process and the inferences students draw from their guided use that leads to conceptual restructuring.** In that sense they function as thought experiments (Kuhn, 1977; Nersessian, 1992) and as analogies (Gentner, 2003). They support the construction of new ideas and provide integration for them. They apply to more than one physical entity.

Once assembled, linchpins “hold new things together” within concepts—the weight measure line embodies and interrelates the extensivity of weight and the principles of weight measurement; a compositional model supports part-whole reasoning. They also hold new things together among concepts (they express a structure common to several entities) and, by doing so, they contribute further to restructuring the knowledge network.

Overall conclusions. To sum up, lever concepts and linchpins not only make sense within a learning progression approach but are entailed by it, as they fulfill the mandate of making sense within the early conceptualization while providing mechanisms for large-scale changes in the

knowledge network. Thus we believe that they will be useful in learning progressions other than ours.

Evaluation: By what criteria can our LPM be evaluated and revised?

LPMs are broad theories (based on existing research) that characterize the evolution of students' knowledge over long periods of time when they successfully move from the lower anchor to upper anchor. These theories not only characterize the lower and upper anchor but also identify what reconceptualizations occur along the way that function as productive stepping stones. At present, these theories are highly conjectural as many students fail to achieve the upper anchor and detailed longitudinal studies have not been done under competing instructional approaches. Yet these proposed learning progressions are important as tools for guiding curriculum and assessment design. Further, as is true for all broad theories, their evaluation is complex and indirect. Only through iterative cycles of testing and evaluating of these more concrete curricular and assessment products can the assumptions underlying the LPM be tested and the LPM itself become better articulated as well as fundamentally revised.

For example, the Inquiry Project has developed an extensive 2-part clinical interview to probe key aspects of students evolving concepts of matter, material, weight, volume, density, and state of matter, along with changing understandings of number, fraction, division, length, area, and abilities to engage with proportional reasoning (see Carraher, Smith, Wiser, & Schleimann, 2009, for more detailed discussion of the assessment instrument and how it was informed by our LPM). Each task focuses on different aspects of conceptual understanding and is open to multiple interpretations by students, so that we could ascertain what properties or relations were salient and important to the child in a given situation. This allowed our analysis to capture the child's meaning and to relate children's patterns of reasoning across different tasks. Each task also entails multiple concepts and involves different levels of difficulty. We are using this interview in an extensive 3-year longitudinal study with 100 students, following these students at multiple points in time as they progress from grade 3 to grade 5. This allows us to examine whether there are systematic patterns in how new understandings emerge, whether there are critical junctures at which new ontological and epistemological insights have clearly emerged, and the extent to which changes in different concepts mutually support each other. A fundamental assumption of our LPM is that progress needs to be made on many aspects and many fronts; at the same time it is not simply piecemeal, but inter-dependent as well as sequential, with certain understandings emerging first and playing a pivotal role in enabling the development of other conceptions. Our assessments not only allow us to test these assumptions, but also to develop more detailed ideas about the nature of these interactions and the usefulness of our assessments in drawing out student ideas. We will learn more about which aspects of our assessments were effective as well as ways the assessment instrument could itself be extended and improved.

An additional way to evaluate the validity of the proposed LPM is to design and implement a curriculum based on it, and test it against curricula based on other learning and teaching theories. Thus, as part of its ongoing three-year longitudinal study, the Inquiry project has also developed and implemented new curriculum units for grade 3 and grade 4 (see Inquiry Project website for details about these curricular units) and is currently designing the unit for

grade 5. More specifically, the longitudinal study is comparing the developing understandings among approximately 60 students experiencing the curriculum developed on the basis of the LPM with approximately 40 students experiencing more traditional curricula (following the prior cohort of students in the same schools, who did not receive the new curricular materials). This allows us to test our assumptions that students in this grade range can be successful in making the conceptual (epistemological, ontological, mathematical) shifts highlighted in our LPM with appropriate curricular support, as well as the assumption that these shifts do not routinely occur for the majority of students with existing (more traditional) elementary school curricula. [Again, the findings of these studies will not only bear on the assumptions of our LP, but provide feedback on ways the curriculum units themselves might be further improved.]

If we find that the new Inquiry curriculum units are more successful than the existing curricula units (used in the schools we were working with) in helping students make progress in our LP, other important questions remain. Most centrally, do these new understandings not only show some coherence but also function as productive stepping stones for later science learning in middle and high school years about the atomic-molecular theory? Further comparative longitudinal studies will be needed to examine these key issues. In addition, it will be instructive to compare the impact of different early (and innovative) instructional approaches (that perhaps have identified different intermediate targets of understanding as useful stepping stones), to compare their long-term effectiveness. We may learn that there are multiple viable paths that are quite different but equally effective or that there are some paths that have better long term payoff. We assume any instructional approach has trade-offs; hence it is important to investigate and understand those trade-offs and down-stream consequences as well as we can.

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